Performability Analysis of Non-repairable Multicomponent Systems Using Order Statistics

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Abstract

Performability, a composite measure that integrates both performance and reliability, has been deemed to be essential in evaluating systems that are capable of trading off performance for reliability under component failures. For non-repairable systems, the goal is to evaluate the distribution or moments of some accumulated reward (performance) defined on a stochastic process that characterizes the different configurations, as successive faults appear. In this paper, instead of assuming that the distributions of sojourn time at various configurations are known, we deduce them from the distributions of the individual component lifetime. We allow lifetimes to be arbitrary but assume that they are independent from one component to another. Knowledge of the reward rates in different configurations allows us to relate the sojourn times to the order statistics of the component lifetimes. We then solve for the distributions of the order statistics. The approach is illustrated in context of a parallel system made of identical components.

1 Introduction

Stochastic modelling and analysis have been extensively used in quantitative evaluation of both the reliability and performance of computer or communication systems. Traditionally, they have been studied as two separate aspects. However, a need for a composite metric for integrating these two arises for analysing the so-called degradable systems. By virtue of their fault detection and reconfiguration capability, they can operate in several degraded modes as a consequence of component failures.

Degradable systems form a special class of fault-tolerant systems where reliability can be traded off at the cost of performance. Performability, a measure that combines both performance and reliability, has been developed as an accepted standard to capture this trade-off [2, 16]. It is defined as the probability that the system reaches an accomplishment level over a utilization interval called the mission time. As faults occur, the system goes through different configurations that can be characterized by a stochastic process. Oftentimes, even if the system is potentially repairable, the designer is concerned with the transient behavior of the system over the mission. Thus the problem reduces to the evaluation of the distribution and/or moments of some accumulated performance metric (reward) defined on the stochastic process. This freedom of choice for the reward makes the measure more powerful and versatile.

The first effort at integrating performance and reliability for a degradable computing system was made by Beaudry [1], where the total amount of computation available from a degrading computer system was analyzed under a Markov model of processor failure. In another early work, Huslende [4] proposed a different measure, the probability that the system performance remains above a certain level during a mission. In the framework of performability introduced by Meyer [2], it was clear that both these measures (and many others) could be interpreted as different assignment of reward rates [8].

Meyer [5] and Fuchtgott and Meyer [6] showed how to obtain closed form solution to the performability distribution by conditioning on the state trajectories (sample paths). We closely follow the outline for analysing systems with arbitrary stochastics of failure that was presented in [6]. Before we present our approach, it should be noted that the integral solutions which arise require enumeration of all possible state trajectories involving numerical algorithms with linear complexity in terms of the trajectories.
For a large system, the computations involved could be very complex depending on the degradation pattern. A semi-Markovian characterization of the underlying stochastic process was considered appropriate for a non-repairable degrading system with independent failures [9], and found to lead to tractable solution [6]. However, as illustrated by Ciardo et al. [12], identifying the sojourn time distribution in the semi-Markov model may not be straightforward for a system with complex configuration.

Due to this complexity, most researchers analyze performability using the Markov reward model [5, 7, 8, 9, 10, 11, 13], where the problem reduces to the transient analysis of the underlying Markov chain. This is basically a generalization of Beaudry’s idea captured in the new framework of performability. However, due to the memoryless property of Markov process, it implies an exponential distribution of sojourn time, and a constant hazard rate. In practice, system components have been better characterized by varying hazard rate over the lifetime, often modelled by Weibull distribution [17]. Although use of exponential approximation to non-exponential distribution may be sufficient for a steady-state analysis [17], it can introduce considerable error in transient analysis [12].

The definition of performability we use is derived from Meyer [2]. The system we consider here has been studied earlier in [5] and [8], where the underlying stochastic process was assumed to be Markov, giving rise to the memoryless distribution of sojourn times at all degraded configurations. The method outlined in this paper is a more general in this regard. We deduce the distributions of the sojourn time from the distributions of individual component lifetimes, that are assumed to be independent from one component to another. The lifetimes need not be memoryless; they can be arbitrary. A semi-Markovian process automatically arises out of the order statistic formulation, but there is no need for explicitly defining its sojourn time distribution.

The rest of the paper is organized as follows. The definition of performability and related measures is presented in section 2, along with the related assumptions and an example. In section 3, we formulate the performability problem for a system with no critical component in terms of the order statistics of random variables representing component lifetimes, and derive closed form solutions for the related measures. These results are then extended to the system with critical component in section 4. In section 5, some numerical values for the derived expressions are plotted to illustrate the effect of the critical component and its mean lifetime. We conclude the paper in section 6 by summarizing the contribution with a focus to further work.

2 Overview

Let \( \{X_t\}_{t \geq 0} \) be a continuous-time discrete-space stochastic process representing the state of the degrading system. The state-space \( Q \) is finite for any practical system, however large it may be. Each state \( q \in Q \) represents a possible configuration of the system with some components failed, and has an associated reward rate \( \rho_q = \rho(q) \), a non-negative real number reflecting the level of performance per unit time, when the system is in state \( q \). The system we analyze here is made of \( n \) identical components, and it degrades gracefully when the components fail one by one. The states are \( q_k, k = 0, \ldots, n \), where \( q_k \) denotes the state with \( k \) faulty components.

Performability of the system over a mission time \( t \) is defined as the probability density function of the accumulated reward (performance)

\[
Y_t = \int_0^t \rho(X_s) \, ds.
\]

The cumulative distribution function of \( Y_t \) is often referred to as the performability distribution function. The mean value of \( Y_t \), which we would call mean reward in mission (MRIM), is a useful characterization.

For a nonrepairable system, the transitions between configurations take the shape of an acyclic graph, so that reentry to a state is not feasible. The sojourn time or the residence time \( \tau_q \), which is the total time spent by the system in configuration \( q \) over the mission time \( t = \sum_{q \in Q} \tau_q \), is therefore contiguos. Equation (1) thus reduces to

\[
Y_t = \sum_{q \in Q} \rho_q \tau_q,
\]

which also provides a simple expression for MRIM

\[
E[Y_t] = \sum_{q \in Q} \rho_q E[\tau_q].
\]

In this paper we focus our attention to the accumulated reward until the complete system failure, and consider the value of \( Y_t \) as \( t \) approaches infinity. The corresponding measures we consider are the distribution function and mean value of

\[
Y_\infty = \lim_{t \to \infty} Y_t = \int_0^\infty \rho(X_t) \, dt,
\]

which we would henceforth call mean reward until failure (MRUF).

Example: Let us consider a single component system having just the working and failed states. If we assign \( \rho = 1 \) for the fault-free state and \( \rho = 0 \) for the failed one, the performability distribution function reduces to \( \Pr[Y_\infty \leq y] = \Pr[T \leq y] \), where \( T \)
is the random variable representing the lifetime of the component. The reliability \( R(t) \) can be written as \( R(t) = \Pr[T > t] = 1 - \Pr[T \leq t] = 1 - \Pr[Y \leq t] \). Clearly, the MRUF reduces to the mean time to failure (MTTF) \([17]\).

One implicit assumption behind formulating a composite performance-reliability measure as above is that the performance and failure rates are functions of only the system state. This means that increasing load or stress during the degradation due to component failure does not affect the lifetime distribution of the surviving components. This means that increasing load or stress has a feature which makes the analysis tractable, but also a desired characteristic in many implementations.

3 Order Statistics Formulation without Critical Component

In the previous section we identified the state-space associated with the graceful degradation of an \( n \)-component system. We start this section with the assumption that all the states \( q_0, k = 0, \ldots, n \) are visited in that strict order as the system degrades gracefully. In effect, we ignore the possibility of any abrupt failure caused by the failure of any critical component that holds the \( n \) components together. The boundaries of the sojourn times can now be identified with the time of failure of the successive components. These are precisely the order statistics of the random variables representing lifetime of the components. In subsection 3.1, we formally introduce the order statistics, and explore how their distributions relate to the distributions of the component lifetimes. Subsection 3.2 shows the result from a traditional Markov model of the order statistics variables to be a special case of these formulations. Finally in subsection 3.3, the closed form expressions for the performatibility distribution and MRUF are derived in terms of the distributions of the order statistics of component lifetimes.

3.1 Order statistics of lifetime

If a family of random variables \( T_1, T_2, \ldots, T_n \) are arranged in ascending order of magnitude and renamed as \( T_{1:n} \leq T_{2:n} \leq \cdots \leq T_{n:n} \), the new random variable \( T_k:n \) is called the \( k \)th order statistic for \( k = 1, 2, \ldots, n \). Suppose \( T_1, T_2, \ldots, T_n \) represent the lifetimes of the identical components, and as a result, they are iid (independent and identically distributed) with cdf \( F(t) = \Pr[T \leq t] \). The cdf \( F_{k:n}(t) \) of the order statistics \( T_{k:n} \) of lifetimes can be derived as

\[
F_{k:n}(t) = \Pr[T_{k:n} \leq t] = \Pr[k \text{ or more components fail in } [0, t]] = \sum_{i=k}^{n} \Pr[\text{Exactly } i \text{ components fail in } [0, t]] = \sum_{i=k}^{n} \binom{n}{i} F(t)^i (1 - F(t))^{n-i}.
\]

The binomial form can also be expressed as an incomplete beta function,

\[
F_{k:n}(t) = I_{F(t)}(k, n - k + 1) = \int_0^t f(t) u^{k-1} (1-u)^{n-k} du / B(k, n - k + 1).
\]

To derive the pdf \( f_{k:n}(t) \), let us consider the probability of the event that the \( k \)th failure occurs in \([t, t+\delta t]\), where \( \delta t \) can always be chosen in such a way that only one failure occurs in this interval. This means \( k - 1 \) components must fail in \([0, t]\), and \( n - k \) must fail in \([t + \delta t, \infty)\). Moreover, there are \( n!/(k-1)!(n-k)! \) ways to choose the first \( k - 1 \), the \( k \)th and the rest \( n - k \) components out of \( n \). As all components fail independently, we have

\[
\Pr[t \leq T_{k:n} \leq t + \delta t] = \frac{n!}{(k-1)!(n-k)!} \{F(t)\}^{k-1} (1 - F(t))^{n-k} f(t) dt.
\]

The same expression can also be obtained by differentiating \( F_{k:n}(t) \) in equation (6) with respect to \( t \).

A straightforward extension of the above argument yields an expression for the joint pdf of two or more order statistics of the lifetime distribution of individual components \([3, 14]\). For the joint pdf \( f_{n_1, \ldots, n_k}(t_1, \ldots, t_k) \) of any \( k \) order statistics \( 1 \leq n_1 < n_2 < \cdots < n_k \leq n \) and \( t_1 \leq t_2 \leq \cdots \leq t_k \), we consider the event that the \( n_k \)th failure occurs in \([t_k, t_k + \delta t_k]\). The first \( n_1 - 1 \) failures are in \([0, t_1]\), \( n_2 - n_1 - 1 \) in \([t_1 + \delta t_1, t_2]\) and so on, until the remaining \( n - n_k \) failures occur in \([t_k + \delta t_k, \infty)\). The choice of the sequence can be done in \( n!/(n_1-1)! \cdots !(n_k - n_{k-1} - 1)! \cdots !\) ways. Due to the independence of their lifetime

\[
\Pr[t_1 \leq T_{n_1:n} \leq t_1 + \delta t_1, \ldots, t_n \leq T_{n_k:n} \leq t_k + \delta t_k] = \frac{n!}{(n_1-1)! \cdots (n_k - n_{k-1})! n!} \{F(t_1)\}^{n_1-1} f(t_1) \delta t_1 \cdots \{F(t_k) - F(t_k+\delta t_k)\}^{n_k-1} \delta t_k \{1 - F(t_k + \delta t_k)\}^{n-n_k}.
\]
which gives
\[
\begin{align*}
\frac{f_{n_1,\ldots,n_k}(t_1,\ldots,t_k)}{n!} &= \frac{\prod_{i=1}^{n_k} f(t_i) \prod_{j=1}^{n_k} \prod_{i=j+1}^{n_k} (1 - F(t_j))}{\prod_{i=1}^{n_k} (n_i - n_k - 1)!},
\end{align*}
\]

\[
F(t) = 1 - e^{-\lambda t}
\]

The transition probabilities are given by
\[
\begin{align*}
P_{k+1} &= (n-k)\Delta t, \quad k = 0, \ldots, n-1 \\
P_{k} &= 1 - (n-k)\Delta t, \quad k = 0, \ldots, n-1 \\
P_{n,n} &= 1 \\
P_{i,j} &= 0 \quad \text{otherwise}
\end{align*}
\]

Denoting \( Pr[T_{kn} \leq t] \) by \( p_k(t) \), for \( k = 1, \ldots, n \), we have
\[
p_k(t + \Delta t) = Pr[T_{kn} \leq t + \Delta t]
\]
\[
= Pr[k or more components fail in [0, t + \Delta t]]
\]
\[
= Pr[k or more components fail in [0, t]] + Pr[One failure in [t, t + \Delta t]] \times Pr[Exactly k - 1 components fail in [0, t]]
\]
\[
= Pr[T_{kn} \leq t] + \left( Pr[T_{k-1,n} \leq t] - Pr[T_{kn} \leq t] \right)
\]
\[
(n-k+1)\Delta t
\]
\[
= p_k(t) + \left( p_{k-1}(t) - p_k(t) \right) \quad \text{from which we obtain}
\]
\[
p_k(t) = \lim_{\Delta t \to 0} \frac{p_k(t + \Delta t) - p_k(t)}{\Delta t}
\]
\[
= (n-k+1)\left( p_{k-1}(t) - p_k(t) \right)
\]

Our formulation of section 3.1 is of course more general in that it can accommodate arbitrary distribution. We now show that by setting \( F(t) = 1 - e^{-\lambda t} \) equation (11) can be derived from equations (5) and (6).

3.2 Comparison with a Markov model

Before evaluating performability or MRUF, let us verify that we have indeed arrived at a more general solution of sojourn time boundaries than what we could expect by transient analysis of Markov model of the system. With Markovian assumptions, the chain of transitions translates to a pure death process with a constant hazard rate \( \lambda \). At most one transition due to failure can occur in the small interval \([t, t + \Delta t]\). This gives rise to exponentially distributed component lifetime with hazard rate \( \lambda \).

\[
F(t) = 1 - e^{-\lambda t}
\]

The transition probabilities are given by
\[
p_k(t) = \begin{cases} 
1 & \text{if } k = 0 \\
(n-k+1)\Delta t & \text{if } k = 1, \ldots, n-1 \\
0 & \text{otherwise}
\end{cases}
\]

Denoting \( Pr[T_{kn} \leq t] \) by \( p_k(t) \), for \( k = 1, \ldots, n \), we have
\[
p_k(t + \Delta t) = Pr[T_{kn} \leq t + \Delta t]
\]
\[
= Pr[k or more components fail in [0, t + \Delta t]]
\]
\[
= Pr[k or more components fail in [0, t]] + Pr[One failure in [t, t + \Delta t]] \times Pr[Exactly k - 1 components fail in [0, t]]
\]
\[
= Pr[T_{kn} \leq t] + \left( Pr[T_{k-1,n} \leq t] - Pr[T_{kn} \leq t] \right)
\]
\[
(n-k+1)\Delta t
\]
\[
= p_k(t) + \left( p_{k-1}(t) - p_k(t) \right) \quad \text{from which we obtain}
\]
\[
p_k(t) = \lim_{\Delta t \to 0} \frac{p_k(t + \Delta t) - p_k(t)}{\Delta t}
\]
\[
= (n-k+1)\left( p_{k-1}(t) - p_k(t) \right)
\]

Denoting \( Pr[T_{kn} \leq t] \) by \( p_k(t) \), for \( k = 1, \ldots, n \), we have
\[
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\]
\[
= Pr[k or more components fail in [0, t + \Delta t]]
\]
\[
= Pr[k or more components fail in [0, t]] + Pr[One failure in [t, t + \Delta t]] \times Pr[Exactly k - 1 components fail in [0, t]]
\]
\[
= Pr[T_{kn} \leq t] + \left( Pr[T_{k-1,n} \leq t] - Pr[T_{kn} \leq t] \right)
\]
\[
(n-k+1)\Delta t
\]
\[
= p_k(t) + \left( p_{k-1}(t) - p_k(t) \right) \quad \text{from which we obtain}
\]
\[
p_k(t) = \lim_{\Delta t \to 0} \frac{p_k(t + \Delta t) - p_k(t)}{\Delta t}
\]
\[
= (n-k+1)\left( p_{k-1}(t) - p_k(t) \right)
\]

Our formulation of section 3.1 is of course more general in that it can accommodate arbitrary distribution. We now show that by setting \( F(t) = 1 - e^{-\lambda t} \) equation (11) can be derived from equations (5) and (6).

3.3 Evaluation of performability

We are now ready to evaluate the performability of the \( n \)-component parallel system. Clearly \( T_{k,n} \), \( k = 1, \ldots, n \), are the sojourn time boundaries. Defining \( T_{0,n} = 0 \), the sojourn time at state \( q_k \) is
\[
r_k = T_{k+1,n} - T_{k,n}, \quad k = 0, \ldots, n-1
\]

The accumulated reward in equation (4) can now be written as (see figure 1)
\[
Y_\infty = \sum_{k=0}^{n-1} \rho_k r_k
\]

\[
= \sum_{k=0}^{n-1} \rho_k (T_{k+1,n} - T_{k,n}), \quad T_{0,n} \triangleq 0
\]

\[
= \sum_{k=1}^{n} (\rho_{k-1} - \rho_k) T_{k,n}, \quad \rho_n \triangleq 0
\]

so that
Pr\{Y_\infty \leq y\} = Pr\{\sum_{k=1}^n (\rho_{k-1} - \rho_k) T_{k:n} \leq y\}
\geq \int \cdots \int f_{n-1:n}(t_1, \ldots, t_n) dt_1 \cdots dt_n
\sum_{k=1}^{n-1} (\rho_{k-1} - \rho_k) t_k \leq y
(14)

Similarly, the MRUF reduces to
E[\text{\smaller{\textit{Y}}}_\infty] = E[\sum_{k=1}^n (\rho_{k-1} - \rho_k) T_{k:n}]
= \sum_{k=1}^{n-1} (\rho_{k-1} - \rho_k) E[T_{k:n}]
= \sum_{k=1}^{n-1} (\rho_{k-1} - \rho_k) \int_0^\infty f_{k:n}(t) dt
(15)

4 Effect of Critical Components

Until now we have not considered system failure due to failure of any critical component. The effect of such a component failure is an abrupt system failure — the graceful degradation gets truncated at that point. This phenomenon is very common in practical systems. However, the results derived in the previous section would still apply to systems where the critical component has much higher reliability as compared to the failing components.

To incorporate the effect of critical components, we have to change the expression for the sojourn time by
\tau_k = 0, T_c \leq T_k
T_c - T_{k:n} T_k:n \leq T_c \leq T_{k+1:n}
T_{k+1:n} - T_{k:n} T_k+1:n \leq T_c
(16)
where \tau_k denotes the lifetime of the critical component, distributed with cdf \text{\smaller{\textit{F}}}_c(t) and pdf \text{\smaller{\textit{f}}}_c(t).

It is also quite straightforward to handle multiple critical components. One can lump the effect onto a single critical component of effective lifetime \text{\smaller{\textit{T}}}_c with cdf \text{\smaller{\textit{F}}}_c(t), by considering the hypothetical critical component to be active until the first real critical component failure. Assuming there are \text{\smaller{\textit{n}}} critical components with individual lifetimes \text{\smaller{\textit{T}}}_{1:c}, \text{\smaller{\textit{T}}}_{2:c}, \ldots, \text{\smaller{\textit{T}}}_{\text{\smaller{\textit{n}}}:c} with cdf \text{\smaller{\textit{F}}}_{\text{\smaller{\textit{c}}},i}(t), \text{\smaller{\textit{F}}}_{\text{\smaller{\textit{c}}},2}(t) \cdots \text{\smaller{\textit{F}}}_{\text{\smaller{\textit{c}}},n}(t) (they may as well be different), the effective distribution of critical component lifetime is given by
\text{\smaller{\textit{F}}}_c(t) = 1 - \prod_{i=1}^{\text{\smaller{\textit{n}}}} (1 - \text{\smaller{\textit{F}}}_{\text{\smaller{\textit{c}}},i}(t)),
(17)
and \text{\smaller{\textit{f}}}_c(t) can be obtained by differentiating \text{\smaller{\textit{F}}}_c(t) with respect to time.

Given that \text{\smaller{\textit{T}}} \leq \text{\smaller{\textit{T}}}_{\text{\smaller{\textit{m}}}:c} \leq \text{\smaller{\textit{T}}}_{\text{\smaller{\textit{m}}+1}:c}, \text{\smaller{\textit{Y}}} from equation (4) reduces to (see figure 2)
\text{\smaller{\textit{Y}}}_{\text{\smaller{\textit{m}}}:c} = \sum_{k=0}^{\text{\smaller{\textit{m}}}-1} \rho_k (\text{\smaller{\textit{T}}}_{k+1:c} - \text{\smaller{\textit{T}}}_{k:c}) + \rho_m (\text{\smaller{\textit{T}}}_{m:c} - \text{\smaller{\textit{T}}}_{m-1:c})
= \sum_{k=1}^{\text{\smaller{\textit{m}}}-1} (\rho_{k-1} - \rho_k) T_{k:c} + \rho_m T_c.
(18)
As \text{\smaller{\textit{T}}} \leq \text{\smaller{\textit{T}}}_{\text{\smaller{\textit{m}}}:c} \leq \text{\smaller{\textit{T}}}_{\text{\smaller{\textit{m}}+1}:c} are disjoint, by the axiom of probability
Pr\{\text{\smaller{\textit{Y}}} \leq y\} = Pr\{\text{\smaller{\textit{Y}}} \leq y \cap \{\text{\smaller{\textit{T}}} \leq \text{\smaller{\textit{T}}}_{\text{\smaller{\textit{m}}}:c}\} \}
+ \sum_{m=1}^{\text{\smaller{\textit{m}}}} \Pr\{\text{\smaller{\textit{Y}}} \leq y \cap \{\text{\smaller{\textit{T}}} \leq \text{\smaller{\textit{T}}}_{\text{\smaller{\textit{m}}}:c}\} \}
+ \Pr\{\text{\smaller{\textit{Y}}} \leq y \cap \{\text{\smaller{\textit{T}}} \leq \text{\smaller{\textit{T}}}_{\text{\smaller{\textit{m}}}:c}\} \}
+ \Pr\{\text{\smaller{\textit{Y}}} \leq y \cap \{\text{\smaller{\textit{T}}} \leq \text{\smaller{\textit{T}}}_{\text{\smaller{\textit{m}}}:c}\} \}
5 Numerical Evaluation and Results

To illustrate how the formulations developed here can be used for computing performability distribution and MRUF, we consider a perfectly scalable system made of \( n \) components and a single critical component. This indicates a reward assignment of the form

\[
\rho_k = (n - k)\rho, \quad k = 0, \ldots, n
\]

The analysis only suggests that the sequence \( \rho_0, \rho_1, \ldots, \rho_n \) is monotonically nonincreasing, they could depend on the system states \( q_0, q_1, \ldots, q_n \) in a potentially complex way. On the other hand, a scalable reward assignment can be used as a consistency check on the correctness of numerical routines, by verifying whether or not linearity between number of components and MRUF results.

To obtain the numerical values for performability distribution and MRUF, the series of integrals in equations (14-15) and equations (19-20) need to be evaluated. We use a simple version of multidimensional Gaussian quadrature to perform the integrations numerically in the specified regions. The grid points for the quadrature are generated by recursively using Gauss-Legendre formula [15]. Our experience shows that Gauss-Legendre quadrature with a suitably large value chosen for infinity is more stable and well-behaved for the exponential class of integrands, as compared to a combination of Gauss-Laguerre and Gauss-Legendre quadrature ideally suited for semi-open regions. Fifty points per dimension were used.

For illustrations and comparisons, we assume that the lifetimes of both the critical and non-critical components are modelled by the Weibull distribution,

\[
Pr[T \leq t] = 1 - e^{-(\lambda t)^\alpha}, \quad \text{MTTF} = E[T] = \frac{\Gamma(1 + \frac{1}{\alpha})}{\lambda}
\]

where the two parameters \( \lambda \) and \( \alpha \) essentially characterize the lifetime distribution of a component. With \( \alpha = 1.0 \), this reduces to the exponential distribution that corresponds to the Markovian model. We study the effect of the increasing and decreasing hazard rates by using two other values of \( \alpha \), viz. 2.0 and 0.5. However, instead of using the same \( \lambda \) values in those three cases, we use the same values for the component MTTF. The reason for this choice is that component MTTFs are more direct results of reliability measurement. While the MTTF is unchanged, increasing (decreasing) hazard rate causes more faults to occur later (earlier) as compared to the constant hazard rate, thereby enhancing (diminishing) the accumulated reward. In each case, we also observe the effect of using more reliable (i.e. higher MTTF) critical components with no change in the quality of non-critical components. The necessary marginal and joint density functions of the order statistics have been used to compute the values of the integrands at the necessary grid points.

Figure 3 shows the effect of critical component on the performability distribution function for a two component system, for different \( \alpha \) values. We restrict the number of components to \( n = 2 \), because for higher values, long CPU hours are needed to evaluate the re-
lated integrals of \((n + 1)\) dimensions. For example, a sample run on a Sparcstation-5 required 888.1 seconds user time to compute \(Pr[Y_{\infty}]\) for twenty y-points with two components, whereas 49240.1 seconds were needed for the computation with three components. Clearly, for every additional component, at least a fifty-fold increase in runtime is expected for handling the next dimension of integral.

The unit used for the accumulated reward is component-day, the total throughput available from a component over one day. As the critical components need be more reliable than the non-critical components, we increase the MTTF of the critical component keeping the MTTF of the non-critical components unchanged. The cdf of the accumulated reward of the system, as a result, approaches in limit that of a system with no critical components. The rate at which the performability distribution approaches the limiting case with increase in MTTF of critical component is more for the increasing hazard rate, and less for the decreasing case.

These observations are, however, not limited to systems with two components. Computing MRUF for systems with critical components requires evaluation of only up to 4-dimensional integrals, although the total number of such integral evaluations grows as \(O(n^2)\). As a result, computing MRUF for larger system is still possible whereas computing performability distribution is not. Table 1 shows the complexity factors to compute the MRUF with varying number of components and the user time needed for a set of sample runs on a Sparcstation-5. We have computed the MRUF's of systems with up to eight components, and they have been plotted against the number of components in figure 4 for the three different \(\alpha\) values.

Component-day is also the choice of unit for MRUF. Naturally, with increasing MTTF of critical component, the MRUF of the system approaches the limiting MRUF of the system with no critical component. As we have observed in the case of performability distribution, this is more (less) prominent with increasing (decreasing) hazard rate in comparison to the Markovian or constant hazard rate case. Thus MRUF is a good indicator of performability in that it preserves the trend. This also means that the reliability requirement of the critical component to achieve the best possible performability is less stringent as the hazard rate increases.

6 Conclusions

The usefulness of a closed form analytical solution for any system measure cannot be overemphasized. In this paper, we have developed a methodology based on order statistics to analytically evaluate performability figures for a non-repairable multicomponent parallel system with one or more critical elements. This description indeed characterizes a simple yet broad class of parallel systems. A SIMD computer can be cited as one example where the processing elements (PEs) form the pool of degradable components, whereas the master CPU, the controller and the clock driver can be identified as the critical components. The applicability of the approach is, however, not limited only to the configuration mentioned above. Under suitable bounding arguments, similar degradation steps can be found embedded in more complex parallel processing systems. The scope of extending the formulation exists in some arrays and interconnection networks.

![Figure 4: Mean reward until failure (a) \(\alpha = 1.0\), (b) \(\alpha = 2.0\), (c) \(\alpha = 0.5\)]
Table 1: Cost of MRUF computation

<table>
<thead>
<tr>
<th>Number of components</th>
<th>Number of integrals 2-d</th>
<th>Number of integrals 3-d</th>
<th>Number of integrals 4-d</th>
<th>User time (sec)</th>
</tr>
</thead>
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<tr>
<td>1</td>
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<td>0</td>
<td>0</td>
<td>0.8</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>66.2</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>4</td>
<td>1</td>
<td>1336.3</td>
</tr>
<tr>
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<td>2</td>
<td>6</td>
<td>3</td>
<td>3831.1</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>8</td>
<td>6</td>
<td>7586.9</td>
</tr>
<tr>
<td>6</td>
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<td>10</td>
<td>10</td>
<td>13249.1</td>
</tr>
<tr>
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<td>12</td>
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<td>20500.3</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td>14</td>
<td>21</td>
<td>29478.7</td>
</tr>
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</table>

Our approach breaks the tradition in that it does not call for making assumptions about the sojourn time distributions, whether memoryless or otherwise. The key assumption is that of the independence of component lifetime, which is more tangible to a system designer. Lifetime of a component can have any general (known) distribution, possibly supplied by the manufacturer. This is clearly in contrast to the Markov model, where hazard rates for components are assumed constant. Thus, in a real system with increasing hazard rate, Markovian analysis may come up with over-optimistic estimates, whereas our analysis should be more accurate. The technique can also prove useful to study the effect of the critical component and to do a requirement analysis on its MTTF.

References


